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ABSTRACT

This report is an extension of the Perturbation Method of K. O. Friedrichs, inasmuch as we do not require that the disturbance should satisfy a certain "smallness" condition. More specifically it is shown, that for a wide class of operators the "smallness" condition can be replaced by the following: the point eigenvalues of the disturbed operator are disjoint from the continuous spectrum of the undisturbed operator.

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§1. Introduction

It is well known that the spectrum of an operator is extremely sensitive to perturbations, [15]. Nevertheless, in his papers ([2],[3],[10]) K. O. Friedrichs showed conditions on the disturbance, "smallness and gentleness", which insured that the disturbed and undisturbed operators were unitarily equivalent. Since the gentleness condition is rather delicate we postpone its exact formulation till Section 2.

Friedrichs also gave examples, [2], (section 5), illustrating that if the disturbance violates either the "smallness" or the "gentleness" condition, the disturbed and undisturbed operators are not necessarily equivalent, owing to the appearance of additional point eigenvalues.

It is most natural to ask whether the examples referred to are typical, in the sense that the "worst" that can happen under "gentle" perturbations is the appearance of point eigenvalues. In other words we ask is it true that under "large" but "gentle" perturbations the "continuous parts" of the disturbed and undisturbed operators are equivalent? We do not know the answer to this question; however, we know that the answer to a weaker question is in the negative. This is shown in Appendix II.

We consider the following problem: <u>is it possible to replace the "smallness" condition by another "additional" condition which insures that under a gentle perturbation the continuous parts of the disturbed and undisturbed operators are unitarily equivalent?</u> For the case of an undisturbed operator

with simple, absolutely continuous spectrum the author gave a possible additional condition in an unpublished work, [18].

In this report we shall extend the result of this work to the case of an undisturbed operator of the following type: its continuous part is absolutely continuous of uniform multiplicity and its singular part is of finite rank. Specifically, in Section 3 we shall show the following: if the undisturbed operator is of the above type, the disturbance is of finite rank, it belongs to a particular space of gentle operators and if the point eigenvalues of the disturbed operator are disjoint from the undisturbed spectrum then the continuous parts of the disturbed and undisturbed operators are unitarily equivalent. In Section 4 we extend the results of Section 3 to the case of perturbations satisfying a certain condition of "complete gentleness". This is the condition introduced by O. A. Ladyzhenskaya and L. D. Faddeev, [6]. This extension process will make essential use of the fact that our "particular" space of gentle operators is a one parameter family of such "gentle" spaces and according to Lemma 4.1 the norms of these gentle spaces are closely related to each other. The result of this extension process, our Main Theorem, can be formulated as follows: Let the undisturbed operator be of the above type, let the disturbance belong to a particular space of gentle operators and be completely gentle. Suppose that the point eigenvalues of the disturbed operator are disjoint from the undisturbed continuous spectrum. Then the continuous parts of the disturbed and undisturbed operators are unitarily equivalent.

This Theorem applies, for example, to the "large and self-adjoint" version of the perturbation problem treated in Chapter I, Section 8 of [10]. This problem is closely related to the perturbation problem entering the Lee-model, which in turn is related to perturbation problems of the quantum theory of fields.

This Theorem also applies to certain differential operators. Nevertheless, it is somewhat tedious to establish the validity of our "additional" condition in its present form. Therefore in a future report we shall "improve" the "additional" condition and we shall establish the "improved additional" condition for these differential operators.

Finally let us mention that the "gentleness" condition can be replaced by a different condition in order to draw the following, somewhat weaker conclusion: <u>the absolutely continuous parts of the disturbed and undisturbed operator are unitarily equivalent</u>. More precisely T. Kato has shown [8] that the above statement holds for trace class perturbations. Later S. T. Kuroda introduced a condition, which essentially says that the disturbance is of trace class with respect to the undisturbed operator, and showed that this condition implies the above statement too, [16]. Let us also mention that most likely the Kato-Kuroda condition does not imply the "stability" of the entire continuous spectrum. For, N. Aronszajn [13] gave an example of a Sturm-Liouville problem, which for one boundary condition has a pure point spectrum, while for other boundary conditions it has a singularly continuous spectrum. On the

other hand, it was pointed out by F. Wolf [14] that under "general" conditions a one-dimensional perturbation of the "domain" of an operator "amounts" to a perturbation of rank 1.

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P_σ denotes the projector on the point-eigenspace of A.

The additional conditions that A has no point-eigenvalues enters the Theorem on Small Perturbations.

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Proposition 1 is loosely stated. It should read as follows:

Proposition 1

Suppose that $K \varepsilon (R)_{\theta,\lambda}$. Then to every positive ε and to every number $\tilde{\theta}$, $0 < \tilde{\theta} < \theta$, there is a kernel K_n such that

$$||K-K_n||_{\tilde{\theta},\lambda} < \varepsilon,$$

and the support of K_n is contained in the union of a finite number of rectangles which in turn is contained in $S_\lambda \times S_\lambda$.

In the proof of the proposition the definition of the family of intervals should be replaced as follows: $\{S_i\}$ is the family of disjoint intervals entering the decomposition of the interior of S_λ. Then $S_\lambda = B \cup \bigcup S_i$ where B is the boundary of S_λ. The fact that gentle kernels vanish on B ensures that the proof of the Proposition goes through with such a modification of the definition of the intervals $\{S_i\}$.

§2. The Space of Gentle Operators.

The following space of gentle operators is a slight generalization of the one introduced in [3,10], inasmuch as we allow that the "undisturbed" operator has point eigenvalues;

1). Let A be a not necessarily bounded self-adjoint[1] operator on a seperable Hilbert space $\mathfrak{H}$.

2). Let (R) be a collection of bounded linear operators R on the Hilbert space $\mathfrak{H}$, which forms a Banach space with respect to a gentleness norm $|| \ ||_g$.

3). Let Γ be a bounded linear transformation on (R) to the space of bounded linear operators on $\mathfrak{H}$.

The triplet $[A,(R),\Gamma]$ defines a space of gentle operators if the following three statements hold:

$\underline{G_1}$: For every R in (R)

$$A\Gamma R - (\Gamma R)A + P_\sigma R P_\sigma = R \quad .$$

$\underline{G_2}$: Let R_1 and R_2 belong to (R). Then $R_1\Gamma R_2$, $(\Gamma R_1)R_2$ belong to (R) and

$$||R_1\Gamma R_2||_g \leq ||R_1||_g ||R_2||_g$$

$$||(\Gamma R_1)R_2||_g \leq ||R_1||_g ||R_2||_g \quad .$$

$\underline{G_3}$: $\Gamma(R_1\Gamma R_2 + (\Gamma R_1)R_2) + P_\sigma \Gamma R_1 \Gamma R_2 P_\sigma = (\Gamma R_1)(\Gamma R_2)$.

Such a space of gentle operators was introduced by Friedrichs, [3,10], in order to insure the validity of a theorem of the following type:

THEOREM ON SMALL PERTURBATIONS.

<u>Let</u> A <u>be a self-adjoint operator and let the family of operators</u> B_n <u>be gentle with respect to</u> A. <u>Suppose that</u> $||B_n||_g \to 0$, <u>as</u> $n \to \infty$. <u>Then for large enough</u> n <u>the operators</u>

A and $A+B_n$ <u>are unitarily equivalent; moreover there is a unitary transformation</u> U <u>of the form</u> $U = I + \Gamma R_n$ <u>with gentle</u> R_n <u>carrying</u> $A+B_n$ <u>into</u> A.

The concept of gentleness defined previously is too general and therefore we introduce simplifying assumptions on A. In order to do this we need a notion introduced by T. Kato in [8]: Let A be a self-adjoint operator with corresponding spectral resolution $E(\lambda)$. The continuous (absolutely continuous, resp. singular) part of the operator A is the restriction of A to the set of all those vectors f, for which $\langle E(\lambda)f,f \rangle$ is a continuous (absolutely continuous resp. singular) function of λ. We use this notion to state:

<u>Condition 2.1</u>

a) The continuous part of A is absolutely continuous and it is of uniform spectral multiplicity.

b) The singular part of A is of finite rank.

c) The point spectrum and the continuous spectrum of A are disjoint.

It is easily seen from the Spectral Representation Theorem (cf. Appendix I) that condition 2.1 is equivalent to the following:

<u>Condition 2.2</u>

The operator A is unitarily equivalent to the multiplication operator on an $\mathcal{L}_2(\mu, \{\mathfrak{H}\})$ -space and the support of the continuous and singular parts of μ are disjoint[(2)]. Moreover $\mathcal{L}_2(\mu, \{\mathfrak{H}\})$ can be written in the form

$$\mathcal{L}_2(\mu, \{\mathfrak{H}\}) = \mathcal{L}_2(\lambda, \mathfrak{H}) \oplus \mathcal{L}_2(\sigma, \{\mathfrak{H}\}), \tag{2.1}$$

where λ is the restriction of the Lebesgue measure to some closed set and

$$\dim \mathcal{L}_2(\sigma, \{\mathfrak{H}\}) < \infty \quad . \tag{2.2}$$

In particular condition 2.2 requires that the family of Hilbert spaces $\{\mathfrak{H}\}$ entering the spectral representation of the continuous part of A should consist of a single Hilbert space $\mathfrak{H}$. Next, we consider special gentle spaces. First, however, we introduce some notations: M_μ denotes the multiplication operator on $\mathcal{L}_2(\mu, \{\mathfrak{H}\})$, i.e. $M_\mu f(x) = xf(x)$. I_μ denotes the identity operator on $\mathcal{L}_2(\mu, \{\mathfrak{H}\})$, and P_λ, (P_σ), denotes the projector on $\mathcal{L}_2(\lambda, \mathfrak{H})$, $(\mathcal{L}_2(\sigma, \{\mathfrak{H}\}))$. Finally, if a measure is not indicated where it should be indicated then it is understood to be the Lebesgue measure on $(-\infty\ +\infty)$. For example, $\mathcal{L}_2(\mathfrak{H})$ denotes the space of $\mathfrak{H}$ valued functions f with norm

$$||f||^2 = \int_{-\infty}^{\infty} < f(x)f(x) > dx \quad ,$$

and M denotes the multiplication operator on $\mathcal{L}_2(\mathfrak{H})$.

<u>The Space</u> $(R)_\theta$

This is a slight variation of the space introduced in [3,10]. We shall define a slightly different <u>gentleness norm</u> which is equivalent to the original one.

Define the <u>gentleness norm</u> with the aid of a "modified Hölder norm" as follows: Let R be an integral operator, i.e.,

$$Rf(x) = \int R(x,y)f(y)dy \quad .$$

To the kernel R(x,y) assign the kernel

$$(2.3)\qquad \tilde{R}(\tilde{x},\tilde{y}) = R(\tan \tilde{x},\ \tan \tilde{y}),\ -\tfrac{\pi}{2} \le \tilde{x} \le \tfrac{\pi}{2}\ \ -\tfrac{\pi}{2} \le \tilde{y} \le \tfrac{\pi}{2} \quad .$$

Define

$$|\tilde{R}| = \sup_{\tilde{x},\tilde{y}} |\tilde{R}(\tilde{x},\tilde{y})|$$

where $|\tilde{R}(\tilde{x},\tilde{y})|$ denotes the norm of the operator $\tilde{R}(\tilde{x},\tilde{y})$ which acts in the accessory space $\mathfrak{H}$.

Let

$$\Delta_{h_1}\tilde{R}(\tilde{x},\tilde{y}) = [\tilde{R}(\tilde{x}+h_1,\tilde{y}) - \tilde{R}(\tilde{x},\tilde{y})]h_1^{-\theta}$$

$$\Delta_{h_2}\tilde{R}(\tilde{x},\tilde{y}) = [\tilde{R}(\tilde{x},\tilde{y}+h_2) - \tilde{R}(\tilde{x},\tilde{y})]h_2^{-\theta} \quad .$$

Then set

$$(2.4)\qquad ||R||_\theta = |\tilde{R}| + \sup_{h_1>0} |\Delta_{h_1}\tilde{R}| + \sup_{h_2>0} |\Delta_{h_2}\tilde{R}|$$

$$+ \sup_{h_1>0,h_2>0} |\Delta_{h_1}\Delta_{h_2}\tilde{R}| \quad .$$

Define $(R)_\theta$ to be the space of those kernels R for which $||R||_\theta < \infty$, and for which

$$(2.5)\qquad \lim_{|x|\to\infty} R(x,y) = \lim_{|y|\to\infty} R(x,y) = 0 \quad .$$

Next define

$$(2.6)\qquad \Gamma R = \lim_{\varepsilon\to 0} \Gamma_\varepsilon R$$

where

$$\Gamma_\varepsilon R(x,y) = \frac{R(x,y)}{x-y+i\varepsilon} \quad .$$

It was shown in [3] that for kernels R in $(R)_\theta$ the Γ transformation is well defined, moreover the triplet $[M,(R)_\theta,\Gamma]$ defines a space of gentle operators.

The Space $(R)_{\theta,\lambda}$.

This is the space introduced in [10] in connection with "Perturbation of Operators with Restricted Spectra".

Recall that λ is the restriction of the Lebesgue measure to some closed set. Let R be an integral operator on $\mathcal{L}_2(\lambda, \mathfrak{H})$; then $R(x,y)$ is defined in $S_\lambda \times S_\lambda$, where S_λ is the support of λ. Extend $R(x,y)$ to the entire plane by setting $R(x,y) = 0$ outside $S_\lambda \times S_\lambda$. Let R_λ denote the extended kernel; then define

$$||R||_{\theta,\lambda} = ||R_\lambda||_\theta \quad .$$

By virtue of the fact that Γ maps operators in $(R)_{\theta,\lambda}$ into bounded operators on $\mathcal{L}_2(\lambda, \mathfrak{H})$, the gentle space defined by the triplet $[M_\lambda, (R)_{\theta,\lambda} \Gamma]$ is a gentle "subspace" defined by the triplet $[M, (R)_\theta, \Gamma]$. Hence the triplet $[M_\lambda, (R)_{\theta,\lambda} \Gamma]$ defines a gentle space of operators.

The Space $(R)_{\theta,\mu}$

This is a slight generalization of the space introduced in [10] in connection with the "Perturbation of an Operator with Point and Continuous Spectrum".

Recall that $\mathcal{L}_2(\mu, \{\mathfrak{H}\})$ satisfies Condition 2.2 and that we denoted by P_λ, (P_σ) the projector on $\mathcal{L}_2(\lambda, \mathfrak{H})$, $(\mathcal{L}_2(\sigma\{\mathfrak{H}\}))$. Let R be a bounded operator on $\mathcal{L}_2(\mu, \{\mathfrak{H}\})$; then clearly

$$R = P_\lambda R P_\lambda + P_\lambda R P_\sigma + P_\sigma R P_\lambda + P_\sigma R P_\sigma \quad .$$

Set

(2.8) $$||R||_{\theta,\mu} = ||P_\lambda R P_\lambda||_\theta + ||P_\lambda R P_\sigma||_\theta + ||P_\sigma R P_\lambda||_\theta \quad .$$

Note that the kernel of $P_\lambda R P_\tau$ is a function of one variable only. In the definition of $||P_\lambda R P_\tau||_\theta$, (2.4), we set the expressions referring to the "other" variable equal to 0. Also note that $P_\sigma R\, P_\sigma$ does not enter the definition of $||R||_{\theta,\mu}$. Next we set

$$(2.9) \qquad \Gamma R = \lim_{\varepsilon \to +\infty} \Gamma_\varepsilon \, [R - P_\sigma R\, P_\sigma] \quad ,$$

and define $(R)_{\theta,\mu}$ to be the space of those kernels R for which $||R||_{\theta,\mu} < \infty$ and which vanish at infinity in the sense of (2.5). Then an argument quite analogous to the one given in [3] shows that for kernels R in $(R)_{\theta,\mu}$ the Γ transformation is well defined, moreover the triplet $[M_\mu, (R)_{\theta,\mu}, \Gamma]$ defines a space of gentle operators. This will be our standard gentle space after imposing the condition $1/2 < \theta < 1$.

§ 3. Gentle Perturbations of Finite Rank.

Throughout this section we assume that K is a self-adjoint operator of finite rank on $\mathcal{L}_2(\mu\{\Omega\})$ and that $K \varepsilon (R)_{\theta,\mu}$ with $\frac{1}{2} < \theta < 1$. Although this restriction on θ does not enter the Theorem on Small Perturbations, we shall make essential use of it. A slightly different version of the example of Appendix II shows that without this restriction our Theorems fail to be true. Hence the condition on θ can not be removed, although we shall weaken it in a future report. For brevity, we shall refer to the condition $K \varepsilon (R)_{\theta,\mu}$ with $\frac{1}{2} < \theta < 1$, as K being gentle.

It is convenient to use the "dyadic" form for operators of finite rank. The dyadic product of two vectors k and ℓ is the linear operator defined by

$$(k >< \ell)f = < \ell, f > k \quad .$$

According to the Spectral Theorem for self-adjoint operators of finite rank, K may be written in the form

$$K = \sum_{i=1}^{N} \varepsilon_i k_i >< k_i \tag{3.1}$$

where the $\{k_i\}$ are the eigenvectors of K and the $\{\varepsilon_i\}$ are the corresponding eigenvalues. Since by assumption K is gentle, the range of K is spanned by gentle functions and hence the eigenfunctions are gentle functions. In other words,

$$k_i \ \varepsilon \ (R)_{\theta,\mu} \qquad i = 1,2,\ldots N \quad . \tag{3.2}$$

Next we consider:

Solution of the Friedrichs' Equation.

We seek a gentle operator R, $R : \mathcal{L}_2(\mu, \{\mathfrak{H}\}) \to \mathcal{L}_2(\lambda, \mathfrak{H})$, which satisfies the Friedrichs' equation;

$$(3.3) \qquad (P_\lambda + \Gamma R)K = R \quad .$$

First we ask for a necessary condition on R. Insertion of (3.1) into (3.3) yields;

$$(P_\lambda + \Gamma R)\sum_{i=1}^{N} \varepsilon_i k_i >< k_i = \sum_{i=1}^{N}(P_\lambda + \Gamma R)k_i >< \varepsilon_i k_i = R \quad .$$

Hence R is of finite rank and it is of the form

$$(3.4) \qquad R = \sum_{i=1}^{N} h_i >< \varepsilon_i k_i \quad ;$$

and the $\{h_i\}$ satisfy the following equations;

$$(3.5) \qquad (P_\lambda + \Gamma[\sum_{j=1}^{N} h_j >< \varepsilon_j k_j])k_i = h_i \quad i = 1,2,\ldots,N \quad .$$

Conversely, let $\{h_i\}$ be a gentle set of functions satisfying (3.5), i.e.,

$$(3.5)_x \qquad (P_\lambda + \Gamma[\sum_{j=1}^{N} h_j >< \varepsilon_j k_j])k_i(x) = h_i(x)$$

holds for every x in S_λ, the support of λ, and $h_i(x) = 0$ otherwise. Then the operator R defined by (3.4) is gentle, it is a solution to the Friedrichs' Equation (3.3) and $R : \mathcal{L}_2(\mu, \{\mathfrak{H}\}) \to \mathcal{L}_2(\lambda, \mathfrak{H})$.

We elaborate on (3.5). Multiplying the i-th equation by ε_i, we obtain the following system of linear equations:

$$(3.6) \qquad [A(x)-E]h(x) = -EP_\lambda k(x) \quad ,$$

where

$$A(x) = \{a_{ij}(x)\}$$

$$a_{ij}(x) = \varepsilon_i\varepsilon_j[\int \frac{\langle k_i(y)k_j(y)\rangle}{x-y}\, d\mu(y) + i\pi \langle k_i(x)k_j(x)\rangle]$$

(3.7)

$$E = \{\varepsilon_{ij}\} \qquad \varepsilon_{ij} = \begin{cases} 0 & i \neq j \\ \varepsilon_i & i = j \end{cases}$$

$$k(x) : \{k_i(x)\}$$

$$h(x) : \{h_i(x)\},$$

k(x) and h(x) are elements of the N-orthogonal sum of $\mathfrak{H}$, which we denote by $\mathfrak{H}^{(N)}$, and $k_i(x)$, $h_i(x)$ are the projections on the i-th component of $\mathfrak{H}^N$.

Note that (3.6) is an equation on $\mathfrak{H}^N$, A(x) is a map of $\mathfrak{H}^N$ into $\mathfrak{H}^N$, and the associated matrix is a numerical valued matrix. Clearly if the numerical valued matrix A(x) - E has an inverse in the ring of matrices then so does the associated map A(x) - E of $\mathfrak{H}^{(N)}$ into $\mathfrak{H}^{(N)}$. In the next Lemma we shall give conditions on the operator $M_\mu + K$ which will ensure the existence of the inverse of the matrix A(x) - E. At the moment let us restrict ourselves to the observation that if the map (A(x)-E) has an inverse then $(A(x)-E)^{-1}$ satisfies the Hölder condition in x. This is clear from

$$(3.8) \quad (A(x_2)-E)^{-1} - (A(x_1)-E)^{-1} = (A(x_2)-E)^{-1} (A(x_2) - A(x_1))(A(x_1)-E)^{-1} \quad .$$

For, in this expression the middle term, considered as an operator on Z^N, the N dimensional complex Euclidean space, satisfies the Hölder condition in the operator norm, since by (3.7) the matrix elements do[3]. On the other hand, according to Appendix III, the matrix $A(x) - E$ tends to the matrix E, as $|x|$ tends to infinity. Hence $|(A(x)-E)^{-1}|$, the norm of the operator $(A(x)-E)^{-1}$ on Z^N, is uniformly bounded provided that this inverse exists. In view of (3.8) these two facts establish that $(A(x)-E)^{-1}$ satisfies the Hölder condition for those values of x for which it exists. In the next Lemma we shall show that if the point eigenvalues of $M_\mu+K$ are disjoint from the continuous spectrum of M_μ, i.e., the support of λ, S_λ, then $(A(x)-E)^{-1}$ exists for every x in S_λ. Clearly this will imply the existence of gentle solutions $\{h_i\}$ to (3.5).

LEMMA 3.1 Suppose that $x_o \varepsilon S_\lambda$, and the map induced by the numerical values matrix $(A(x_o)-E)$ is not one-to-one on Z^n. Let $\underline{a}$ be a non zero vector for which

$$(A(x_o)-E)\underline{a} = 0 \quad ; \tag{3.9}$$

then

$$g(x) = \frac{\varepsilon_i k_i(x)a_1+\ldots+\varepsilon_N k_N(x)a_N}{x_o-x}$$

is an eigenfunction of the operator $M_\mu+K$ with the eigenvalue x_o.

First, let us show that $g(x)$ is in $\mathfrak{L}_2(\mu, \{\mathfrak{H}\})$. In order to do this, it suffices to verify the relation

$$\varepsilon_i k_i(x_o)a_1+\ldots+\varepsilon_N k_N(x_o)a_N = 0 \quad . \tag{3.10}$$

For, if it holds we have

$$g(x) = \frac{\varepsilon_1(k_1(x)-k_1(x_o)a_1 + \varepsilon_N(k_N(x)-k_N(x_o))a_N}{x_o-x}$$

and this function is in $\mathcal{L}_2(\mu,\{\mathfrak{H}_i\})$ since the $k_i(x)$ $i=1,\dots,N$ are in $(R)_{\theta,\mu}$ and $\frac{1}{2} < \theta < 1$. The proof of (3.10) is based on the observation that the quadratic form associated with $A(x) - E$ on Z^n can be written as the sum of a real quadratic form on Z^n and i-times the square of the norm of a vector in $\mathfrak{H}$. More precisely the following holds:

(3.11) $$< a(A(x)-E)a > = < a(B(x)-E)a > + i\pi|\varepsilon_1 k_1(x)a_1 + \varepsilon_N k_N(x)a|_N|^2$$

where

$$B(x) = \{b_{ij}(x)\}$$

$(3.11)_b$ $$b_{ij}(x) = \varepsilon_i\varepsilon_j \int \frac{<k_i(y)k_j(y)>}{x-y}\, d\mu(y) \quad .$$

Since $B(x)$ is a Hermitian matrix the quadratic form $< a(B(x)-E)a >$ is real. By assumption $(A(x)-E)a = 0$, hence

(3.12) $$< a(B(x)-E)a > + i\pi|\varepsilon_1 k_1(x)a_1 + \dots+\varepsilon_N k_N(x)a_N|^2 = 0 \quad .$$

In this equation the first term is real, the second term is purely imaginary; hence both terms are equal to zero. This proves (3.10) and the statement that $g(x)$ is in $\mathcal{L}_2(\mu,\{\mathfrak{H}\})$.

It remains to verify the relation

(3.13) $$(M_\mu+K)g = x_o g \quad .$$

In view of (3.7)$(3.11)_b$ and (3.9) we have

$$(3.14)\quad (B(x)-E)a+i\pi\begin{pmatrix}\langle\varepsilon_1\varepsilon_1k_1(x)k_1(x)\rangle\ldots\langle\varepsilon_1\varepsilon_Nk_1(x)k_N(x)\rangle\\ \langle\varepsilon_N\varepsilon_1k_N(x)k_1(x)\rangle\ldots\langle\varepsilon_N\varepsilon_Nk_N(x)k_N(x)\rangle\end{pmatrix}\begin{pmatrix}a_1\\ a_N\end{pmatrix}=0$$

Now the left side of (3.14) can be written in the form

$$(B(x)-E)a+i\pi\begin{pmatrix}\langle\varepsilon_1k_1(x),\ \varepsilon_1k_1(x)a_1+\ldots+\varepsilon_Nk_N(x)a_N\rangle\\ \langle\varepsilon_Nk_N(x),\ \varepsilon_1k_1(x)a_1+\ldots+\varepsilon_Nk_N(x)a_N\rangle\end{pmatrix}$$

Insertion of (3.10) in this relation shows that the vector in ( ) is the 0 vector. Hence from (3.14) we deduce the important relation:

$$(3.15)\qquad (B(x)-E)\underline{a} = 0 \quad .$$

Next we establish relation (3.13) as a corollary of (3.15). Clearly

$$Kg(x) = \sum_{i=1}^{N}\varepsilon_ik_i(x)\ \langle\ k_ig\ \rangle$$

$$= \sum_{i=1}^{N}\varepsilon_ik_i(x)\ \sum_{j=1}^{N}\int\frac{\langle k_i(y),\varepsilon_jk_j(y)\rangle a_j}{x_o-y}\,d\mu(y)$$

$$= k_1(x)(B(x_o)\underline{a})_1+\ldots+k_N(x)(B(x_o)\underline{a})_N \quad .$$

On the other hand, according to the definition of g

$$M_\mu g(x) = xg(x) = \frac{x}{x_o-x}(\varepsilon_1k_1(x)a_1+\ldots+\varepsilon_Nk_N(x)a_N)$$

$$= (\frac{x_o}{x_o-x} - 1)(\varepsilon_1k_1(x)a_1+\ldots+\varepsilon_Nk_N(x)a_N)$$

$$= x_og(x) - \varepsilon_1k_1(x)a_1-\ldots-\varepsilon_Nk_N(x)a_N \quad .$$

Therefore,

$$(M_\mu+K)g(x) = k_1(x)(B(x_o)\underline{a})_1+\ldots+k_N(x)(B(x_o)\underline{a})_N$$

$$- \varepsilon_1 k_1(x)a_1-\ldots-\varepsilon_N k_N(x)a_N + x_o g(x)$$

$$= k_1(x)((B(x_o)-E)\underline{a})_1+\ldots+k_N(x)((B(x_o)-E)\underline{a})_N$$

$$+ x_o g(x) \quad .$$

Using (3.15) we obtain

$$(M_\mu+K)g(x) = x_o g(x)$$

which proves (3.13) and completes the proof of Lemma 3.1.

In view of this Lemma and the remark made before it, we have established the following:

THEOREM 3.1

Let the space $\mathcal{L}_2(\mu, \{\mathfrak{H}\})$ *satisfy Condition 2.2 and let* K *be a gentle self-adjoint operator of finite rank on this space. Suppose that the point eigenvalues of* $M_\mu+K$ *are disjoint from the continuous spectrum of* M_μ. *Then for the operator* K, *the Friedrichs' Equation, (3.3) admits a gentle solution* R.

Having established the existence of such an operator R, we refer to the considerations of [10], which show that one can construct a spectral transformation of the continuous part of $M_\mu+K$ with the aid of this operator R. More precisely, from Theorem 3.1 one can derive the following:

THEOREM 3.2

Let R be a gentle solution of the Friedrichs' Equation (3.3). Then the singular part of $M_\mu+K$ is of finite rank. The continuous part of $M_\mu+K$, $(M_\mu+K)_c$ is unitarily equivalent to M_λ, in particular, it is absolutely continuous. Moreover, $(M_\mu+K)_c$ admits a spectral transformation U of the form $U = I + \Gamma R$. In other words, U is a partial isometry whose initial set is the ortho-complement of the point eigenspace of $M_\mu+K$, its final set is $\mathcal{L}_2(\lambda, \mathfrak{H})$, and

$$(3.16) \qquad (M_\mu+K)_c = U^* M_\lambda U \quad .$$

§4. Arbitrary, Completely Gentle Perturbations.

A) Completely Gentle Operators.

We start this Section by describing the notion of "completely gentle operators" and by showing that: these operators can be approximated in the norm of any "intermediate $(R)_{\theta,\lambda}$ -space" by gentle operators of finite rank. Thus the notion of a completely gentle operator will be a generalization of the notion of a gentle operator of finite rank.

DEFINITION

An operator K is completely gentle if it is gentle, and for every x and y K(x,y) is a compact operator on the accessory space $\mathfrak{K}$.

Using this notion we state:

THEOREM 4.1 (Approximation Theorem)

Let S denote the set of completely gentle, Hermitian symmetric kernels of $(R)_{\theta,\lambda}$. Then the subset of S consisting of kernels of finite rank is dense in S in the norm of any $(R)_{\tilde{\theta},\lambda}$ space with $0 < \tilde{\theta} < \theta$.

Recall that in Section 2 the modified Hölder norm on an unbounded interval was defined with the aid of the ordinary Hölder norm on a bounded interval. Therefore it is no loss of generality to assume that the support of λ is bounded.

The proof of this Approximation Theorem is based on the following simple fact:

LEMMA 4.1

Let λ be the restriction of the Lebesgue measure to a bounded set, and let $\theta_1 < \theta$. Suppose that the sequence K_n tends

to K in the $(R)_{\theta_1,\lambda}$ norm and that K_n remains bounded independently of n in the $(R)_{\theta,\lambda}$ norm. Then K_n tends to K in the norm of any $(R)_{\tilde{\theta},\lambda}$ -space with $\theta_1 \leq \tilde{\theta} < \theta$.

Once the statement of the Lemma is observed its proof is immediate and therefore we do not give the details.

As a first application of this Lemma we show that it is no loss of generality to assume that the support of λ, S_λ, consists of a finite number of intervals: More specifically we shall establish:

Proposition 1.

Let $K \in (R)_{\theta,\lambda}$. Then there is a sequence of kernels K_n such that

$$(4.2) \qquad ||K_n - K||_{\tilde{\theta},\lambda} \longrightarrow 0 \quad , \quad 0 < \tilde{\theta} < \theta \quad ,$$

and

$$K_n \in (R)_{\tilde{\theta},\tilde{\lambda}} \quad ,$$

where the support of $\tilde{\lambda}$ consists of a finite number of intervals.

Since S_λ is a closed set it is the countable union of disjoint closed intervals, $S_\lambda = \bigcup S_i$. We shall also label the intervals, entering the decomposition of S_λ, by points of S_λ. More precisely let S_x denote that one of the above intervals which contains x. Note that since S_λ was assumed to be bonded there are only a finite number of intervals S_x, whose length, $|S_x|$, is greater than $\frac{1}{n}$. Now set

$$(4.3) \qquad K_n(x,y) = \begin{cases} K(x,y) & |S_x| > \frac{1}{n} \text{ and } |S_y| > \frac{1}{n} \\ 0 & \text{either } |S_\lambda| \leq \frac{1}{n} \text{ or } |S_y| \leq \frac{1}{n} \end{cases} \quad .$$

Then clearly

$$(4.4) \qquad ||K_n||_{\theta,\lambda} < 2||K||_{\theta,\lambda}$$

and in view of the fact that K vanishes on the boundary of $S_\lambda \times S_\lambda$, we have that K_n tends to K uniformly in $S_\lambda \times S_\lambda$. Note that this means $(R)_{0,\lambda}$ convergence, hence by virtue of (4.4) we conclude from Lemma 4.1 that

$$(4.2) \qquad ||K-K_n||_{\tilde{\theta},\lambda} \to 0 \quad .$$

Since the support of K_n consists of a finite number of rectangles clearly $R_n \varepsilon (R)_{\tilde{\theta},\tilde{\lambda}}$. This establishes Proposition 1.

For brevity we set $K_n = K$, $\tilde{\theta} = \theta$, $\tilde{\lambda} = \lambda$, and assume that the support of λ consists of a finite number of intervals. As a matter of fact we assume that the support of λ is the interval $[0,\pi]$.

As a next application of Lemma 4.1 we show that K can be approximated by a finite sine series with operator valued coefficients. More specifically let K_n denote the n-th Fejér sum of the double Fourier sine series of K. Then we maintain that

$$(4.5) \qquad ||K_n-K||_{\tilde{\theta},\lambda} \to 0 \quad ,$$

with $0 < \tilde{\theta} < \theta$, and that K_n is Hermitian symmetric if K is. For, as in the case of numerical valued functions of one variable, the Fejér sums of the double Fourier sine series tend uniformly to the function, provided that the odd extension of the function is continuous. Hence the sequence K_n satisfies the first condition of the Lemma with exponent $\theta_1 = 0$. The

second condition (i.e. the fact that $||K_n||_{\theta,\lambda}$ remains bounded independently of n) is seen to be satisfied from the explicit formula for the Fejér sum:

$$K_n(x,y) = \frac{1}{n^2\pi^2}\int_0^{\pi}\int_0^{\pi} K(x+u,y+v)\ \frac{\sin^2 \frac{1}{2}nu\ \sin^2 \frac{1}{2}\,nv}{\sin \frac{1}{2}u\ \sin \frac{1}{2}v}\ dudv \quad .$$

Hence we deduce from Lemma 4.1 the validity of (4.5).

Clearly $K_n(x,y)$ is of the form

$$(4.6) \qquad K_n(x,y) = \sum_{k=1,\ell=1}^{n} A_{k\ell} \sin kx \sin \ell y \quad .$$

In case the accessory space $\mathfrak{K}$ is finite dimensional, we see from (4.6) that the range of K_n is a finite dimensional subspace of $\mathcal{L}_2(\lambda,\mathfrak{K})$, hence K_n is of finite rank.

In case $\mathfrak{K}$ is infinite dimensional we need an additional approximation. The possibility of this approximation hinges on the assumption that K is completely gentle, i.e., K is gentle and K(x,y) is a compact operator on the accessory space $\mathfrak{K}$. In view of this assumption the coefficients $A_{k\ell}$ entering (4.6) are compact operators and therefore there is a sequence of projectors P_m, with finite dimensional range, such that

$$||P_m A_{k\ell} P_m - A_{k\ell}|| \longrightarrow 0 \qquad \text{as } m \longrightarrow \infty \quad .$$

Since according to (4.6) $K_n(x,y)$ is a finite linear combination of the $A_{k\ell}$, we also have;

$$||P_m K_n(x,y) P_m - K_n(x,y)|| \longrightarrow 0 \qquad \text{as } m \longrightarrow \infty \quad .$$

From this fact and from Lemma 4.1 we conclude

(4.7) $\qquad ||P_m K_n P_m - K_n||_{\tilde{\theta},\lambda} \longrightarrow 0 \qquad$ as $m \longrightarrow \infty$.

In view that $P_m K_n P_m$ is a Hermitian symmetric kernel of finite rank and $P_m K_n P_m \in (R)_{\tilde{\theta},\lambda}$, (4.7) completes the proof of the Approximation Theorem.

By virtue of this Theorem every completely gentle perturbation is the sum of a "small" and of a gentle perturbation of finite rank. In Section 2, the Friedrichs' Theorem described the spectral transformation arising from a "small" perturbation. In Section 3, Theorem 3.2 described the spectral transformation arising from a gentle perturbation of finite rank; possibly "large". In the remaining part of this Section we shall show, following [2], how these facts can be used to describe the spectral transformation arising from an arbitrary completely gentle perturbation.

Recall that the space $\mathcal{L}_2(\mu, \{\mathfrak{K}\})$ was assumed to satisfy:

Condition 2.2

The support of the continuous and singular parts of the measure μ are disjoint. Moreover

$$\mathcal{L}_2(\mu, \{\mathfrak{K}\}) = \mathcal{L}_2(\lambda, \mathfrak{K}) \oplus \mathcal{L}_2(\sigma, \{\mathfrak{K}\})$$

where λ is the restriction of the Lebesgue measure to some closed set, and

$$\dim \mathcal{L}_2(\sigma, \{\mathfrak{K}\}) < \infty \quad .$$

Also recall that the following condition enters Theorem 3.2.

Condition 4.1. ("Additional Condition")

The point eigenvalues of $M_\mu+K$ are disjoint from the continuous spectrum of M_μ, i.e. from the support of λ.

Next we state the Main Theorem.

Main Theorem.

Let the space $\mathfrak{L}_2(\mu, \{\mathfrak{H}\})$ *satisfy Condition 2.2 and let* K *be a completely gentle operator in* $(R)_{\theta,\mu}$ *with* $1/2 < \theta < 1$. *Suppose that the operator* $M_\mu+K$ *satisfies Condition 4.1. Then the singular part of* $M_\mu+K$ *is of finite rank. The continuous part of* $M_\mu+K$, $(M_\mu+K)_c$, *is unitarily equivalent to* M_λ, *in particular it is absolutely continuous. Moreover* $(M_\mu+K)_c$ *admits a spectral transformation of the form* $U = P_\lambda + \Gamma R$ *with* $R \in (R)_{\tilde{\theta},\mu}$ *where* $0 < \tilde{\theta} < \theta$.

Proof. Observe that $M_\mu+K$ can be written in the form

$$M_\mu+K = M_\lambda+P_\lambda K P_\lambda + F \tag{4.8}$$

with

$$F = K - P_\lambda K P_\lambda + M_\sigma \tag{4.9}$$

and in view of Condition 2.2 F is of finite rank.

By assumption K is completely gentle and $K \in (R)_{\theta,\mu}$, hence $P_\lambda K P_\lambda$ is completely gentle and $P_\lambda K P_\lambda \in (R)_{\theta,\lambda}$. Therefore from the Approximation Theorem we can conclude the existence of a sequence of gentle operators of finite rank, such that

$$||(P_\lambda K P_\lambda)_n - P_\lambda K P_\lambda||_{\tilde{\theta},\lambda} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty \tag{4.10}$$

where $0 < \tilde{\theta} < \theta$. By virtue of the Friedrichs' Theorem on Small Perturbations (cf. Section 2), (4.10) ensures that for large

enough n there is a unitary transformation W_n: $\mathfrak{L}_2(\lambda, \mathfrak{H}) \to \mathfrak{L}_2(\lambda, \mathfrak{H})$, of the form $W_n = I_\lambda + \Gamma R$, $R \varepsilon (R)_{\theta,\lambda}$, such that

$$(4.11) \qquad M_\lambda + (P_\lambda K P_\lambda)_n - P_\lambda K P_\lambda = W_n^* M_\lambda W_n \quad .$$

Now let us extend W_n to the entire space $\mathfrak{L}_2(\mu, \{\mathfrak{H}\})$ by setting $W_n = I_\sigma$ on $\mathfrak{L}_2(\sigma, \{\mathfrak{H}\})$. Then clearly the extended W_n is unitary and satisfies (4.11). For brevity, from now on we denote by W_n the extended W_n.

Inserting (4.11) in (4.8) and using the unitary character of W_n, yields

$$(4.12) \qquad M_\mu + K = M_\lambda + P_\lambda K P_\lambda + F = W_n^*[M_\lambda + W_n[(P_\lambda K P_\lambda)_n + F]W_n^*]W_n \quad .$$

In other words $M_\mu + K$ is unitarily equivalent to

$$(4.13) \qquad M_\lambda + W_n[(P_\lambda K P_\lambda)_n + F]W_n^* \quad .$$

Since Condition (4.1) is unitarily invariant and $M_\mu + K$ satisfies it, the operator in (4.13) also satisfies Condition 4.1. On the other hand in view of that $W_n = P_\lambda + \Gamma R_n$, $R_n \varepsilon (R)_{\theta,\lambda}$, W_n carries gentle operators into gentle operators. Therefore Theorem 3.2 applies to the operator in (4.13). Insertion of the statement of Theorem 3.2 in (4.12) completes the proof of the Main Theorem.

Footnotes

(1) The Friedrichs' method applies to non self-adjoint problems too. Small and non self-adjoint perturbations are treated in [10], throughout. The connection between the two notations in case of a self-adjoint problem is the following: Our R corresponds to R^+ of [10], and our R^* corresponds to R^- of [10]. Similarly U, U^+, and U^*, U^- are corresponding pairs. Let us also mention that "large" and non self-adjoint perturbations are treated in [11].

(2) If the supports of the absolutely continuous and singular parts of μ are not disjoint, the space $(R)_{\theta,\mu}$ is not a space of gentle operators anymore. Nevertheless the example treated in Section 6 of [3] shows that after suitable modifications the method applies.

(3) We make use of the fact that the Hilbert transform of a Hölder continuous function is Hölder continuous, [cf. 1,5].

Appendix I

The concept of an $\mathfrak{L}_2(S,\sum,\mu,\{\mathfrak{H}\})$ space and the Spectral Representation Theorem.

The space $\mathfrak{L}_2(S,\sum,\mu,\{\mathfrak{H}\})$.

This space is a generalization of the space $\mathfrak{L}_2(S,\sum,\mu,\mathfrak{H})$ introduced in [9], and we define it by stating a slightly more abstract version of the definition of an "ordinary" $\mathfrak{L}_2$ space given in [19].

Let $(S,\sum,\mu)$ be a measure space where in addition S is a metric space with measurable spheres. Let $\{\mathfrak{H}(x)\}$ be a family of Hilbert spaces labeled by elements $x \varepsilon S$, satisfying the following three conditions:

1) the $\{\mathfrak{H}(x)\}$ are subspaces of a given Hilbert space $\mathfrak{H}$, i.e. for every $x \varepsilon S$, $\mathfrak{H}(x) \subset \mathfrak{H}$.

2) the family $\{\mathfrak{H}(x)\}$ is nested in the sense that for every pair (x,y) either $\mathfrak{H}(x) \subset \mathfrak{H}(y)$ or $\mathfrak{H}(y) \subset \mathfrak{H}(x)$.

3) the numerical valued function $\dim \mathfrak{H}(x)$ is μ-measurable.

We start the definition of the integral by introducing a family of "nice and simple" functions as follows:

Definition

A function f on S to $\{\mathfrak{H}\}$ is μ-piecewise constant if it assumes a finite set of values $\{f_i$ only, and the inverse images of the f_i under f are measurable sets.

Next we define an inner product for μ-piecewise constant functions by setting

$$(4) \qquad \langle f,g \rangle = \sum_{i,j} \langle f_i g_j \rangle \mu E_{ij} \quad ,$$

where

$$E_{ij} = \{x | f(x) = f_i, g(x) = g_j\} \ .$$

Clearly the μ-piecewise constant functions form an inner product space, and we call the completion of this inner product space an $\mathfrak{L}_2$ space. Now let us make a formal statement of this:

Definition

$\mathcal{L}_2(S, \sum, \mu, \{\mathfrak{H}\})$ is the completion of the inner product space formed by μ -piecewise constant functions and by the inner product introduced in (4).

Note that we defined this space with the aid of "very special" functions, and we do not need the characterization of those functions which "determine" a Cauchy sequence of piecewise constant functions. All that we need is the fact that under conditions (1) and (2) this is the case for piecewise continuous functions with compact support. In the following we shall make this statement precise and we shall prove it.

Definition

A function f on S to $\{\mathfrak{H}\}$ is μ piecewise continuous if it can be written in the form

$$f = \sum_{n=1}^{N} f_n c_n$$

where the functions $\{f_n\}$ are continuous and the $\{c_i\}$ are characteristic functions of μ -measurable sets.

Lemma

Let f be a μ -piecewise continuous function on S to $\{\mathfrak{H}\}$ with compact support. Then,

a) there is a sequence of μ -piecewise constant functions $\{f_n\}$, which tends to f uniformly on S, and

(5) $$||f_n - f_m|| \longrightarrow 0 \qquad \text{as } n,m \longrightarrow \infty \quad .$$

b) if f_n and $\tilde{f}_n$ are two sequences satisfying statement (a) then

(6) $$||(f_n - \tilde{f}_n) - (f_m - \tilde{f}_m)|| \longrightarrow 0 \qquad \text{as } n,m \longrightarrow \infty \quad .$$

Proof:

We start with part (a). Clearly it suffices to establish the statement for continuous functions f. Let $\varepsilon > 0$ be given. Then in view of the fact that the support of f is compact, it can be covered by a finite number of spheres S_i, with centers x_i, in such a way that

(7) $$|f(x) - f(x_i)| < \varepsilon \qquad \text{for } x \in S_i \quad .$$

Now set

$$f_\varepsilon = \sum f_i c_i$$

where f_i is any continuous function which is $f(x_i)$ on S_i, and c_i is the characteristic function of the intersection of S_i with the support of f. Then clearly f_ε is a μ -piecewise constant function on S to $\mathfrak{H}$ which tends uniformly to f as ε tends to 0. In order to assure that f_ε be an $\{\mathfrak{H}\}$-valued function we make the following adjustment:

Let $\mathfrak{H}_i = \bigcap\limits_{x \in S_i} \mathfrak{H}(x)$, and let P_i denote the projector on $\mathfrak{H}_i$. Then set

(8) $$f_{(\varepsilon)} = \sum P_i f_i c_i$$

We maintain that,

(9) $$\sup_{x \in S_i} |f(x) - f_{(\varepsilon)}(x)| < 2\varepsilon \quad .$$

For, if x is outside the support of f, then according to the definition of the functions $\{c_i\}$ $f_{(\varepsilon)}(x) = 0$. If this is not the case then $x \in S_i$, and in view of condition (2) we have a sequence of projectors P(x), each projecting on $\mathfrak{H}(x)$, such that

(10) $$P_i f(x_i) = \lim P(x) f(x_i) \quad .$$

Now, from (7) we conclude that

$$|P(x)f(x_i) - P(x)f(x)| = |P(x)f(x_i) - f(x)| < \varepsilon$$

hence

$$|P(x)f(x_i) - f(x_i)| < 2\varepsilon \quad ,$$

and therefore, according to (10),

$$|P_i f(x_i) - f(x_i)| < 2\varepsilon \quad .$$

This establishes (9) i.e., the fact that $f_{(\varepsilon)}$ tends uniformly to f. Since the validity of (5) is immediate, part (a) of the Lemma is established. On the other hand part (b) of the Lemma is an immediate consequence of the fact that the measure of a compact set is finite. This completes the proof of the Lemma.

Note that the Lemma does not give any information on the existence of piecewise continuous $\{\mathfrak{H}\}$ valued functions. Nevertheless one could derive from condition (3) that there are "enough" piecewise continuous functions in the following sense. Let $\{\mathfrak{H}_1\}$ and $\{\mathfrak{H}_2\}$ be two families of Hilbert spaces satisfying conditions (1),(2),(3), and in addition let $\mathfrak{H}_1(x) \oplus \mathfrak{H}_2(x) = \mathfrak{H}$, where $\mathfrak{H}$ is independent of x. Then

$$\mathcal{L}_2(S,\Sigma,\mu,\{\mathfrak{H}_1\}) \oplus \mathcal{L}_2(S,\Sigma,\mu,\{\mathfrak{H}_2\})$$
$$= \mathcal{L}_2(S,\Sigma,\mu,\mathfrak{H}) \quad .$$

Remark

The $\mathcal{L}_2(S,\Sigma,\mu,\{\mathfrak{H}\})$ -space is a particular case of a "direct integral-of-Hilbert spaces", introduced in [4,20]. Nevertheless the results of [4,20] indicate that an arbitrary "direct-integral space" is isomorphic to an $\mathcal{L}_2(S,\Sigma,\mu,\{\mathfrak{H}\})$ -space via an isomorphism which preserves the spectral resolution of the corresponding multiplication operators.

In the following we state the Spectral Representation Theorem, which asserts that an arbitrary self-adjoint operator is unitarily equivalent to the multiplication operator on a certain $\mathcal{L}_2(S,\Sigma,\mu,\{\mathfrak{H}\})$ -space. The $\mathcal{L}_2$ spaces entering this theorem can be chosen to have the following two features:

a) $S = [-\infty, +\infty]$

b) $\{\mathfrak{H}\}$ is an increasing family in the sense that $\mathfrak{H}(x) \subset \mathfrak{H}(y)$ if $x < y$.

For brevity we call such a space an increasing $\mathcal{L}_2(\mu,\{\mathfrak{H}\})$ -space. Using this notion we state:

Definition

Let A be an arbitrary operator on an arbitrary Hilbert space $\mathfrak{H}$. If there is a unitary transformation U from $\mathfrak{H}$ to some increasing $\mathcal{L}_2(\mu, \{\mathfrak{H}\})$ -space, such that U carries A into the multiplication operator on $\mathcal{L}_2(\mu, \{\mathfrak{H}\})$, (i.e. $A = U^* M U$) then U is called a spectral transformation of A.

Spectral Representation Theorem.

An arbitrary strictly self-adjoint operator A defined on a seperable Hilbert space admits a spectral transformation.

Appendix II

Example of a gentle operator $K \varepsilon (R)_{\theta,\lambda}$ for which the Friedrichs' equation has no gentle solution.

Let $\theta = \frac{2}{3}$, let λ denote the restriction of the Lebesgue measure to the interval [-2,2], and let $\mathfrak{L}_2(\lambda,\mathfrak{R})$ be the $\mathfrak{L}_2$ space of numerical valued functions. Next set

$$K(x,y) = k(x)k(y) \quad ,$$

where

(1) $$k(x) = \begin{cases} x^{2/3} & |x| \le 1 \\ k(x) & 1 \le |x| \le 2 \end{cases} ,$$

and for $1 \le |x| \le 2$ $k(x)$ is a continuously differentiable function such that

$$k(\pm 1) = 1 \quad , \quad k(\pm 2) = 0$$

$$\int_{1 \le |x| \le 2} \frac{k^2(x)}{x}\, dx + 1 = 0 \quad .$$

Now we maintain that for this kernel the solution of the Friedrichs' equation is discontinuous at $x = 0$. For, according to (3.4),(3.6) and (3.7) the solution of the equation in the case of operators of rank 1 is given by

(2) $$R(x,y) = h(x)k(y) \quad ,$$

with

(3) $$h(x) = \frac{-k(x)}{\int_{-2}^{2} \frac{|k(y)|^2}{x-y}\, dy - 1 + i\pi k\bar{k}(x)}$$

Now clearly

(4) $$\int_{-1}^{+1} \frac{|k(y)|^2}{x-y}\, dy = \int_{-1}^{+1} \frac{y^{4/3}}{x-y}\, dy = \int_{-1}^{+1} \left(\frac{y^{4/3}}{x-y} - \frac{y^{4/3}}{y}\right) dy$$
$$= x \int_{-1}^{+1} \frac{1}{x-y}\, y^{1/3} dy \quad ,$$

and in view of the fact that $y^{1/3}$ is Hölder continuous, so is the second term in (4). On the other hand, according to construction the function

$$\int_{1\leq|y|\leq 2} \frac{|k(y)|^2}{x-y}\, dy - 1$$

vanishes at x = 0, and since it is differentiable it vanishes at least in first order. Therefore the denominator of (3) vanishes at least in first order at x = 0. Since the numerator of (3) vanishes in 2/3 order the quotient in (3) is not continuous at x = 0. In view of (2) this establishes the fact that the solution of the Friedrichs' equation is not continuous, hence not gentle.

According to Theorem 3.1 the operator M + K has a point-eigenvalue in [-2, +2], and according to Lemma 3.1 the function

$$g(x) = \frac{-k(x)}{x}$$

is an eigenfunction of M+K with eigenvalue 0. Of course using the fact that K is of rank 1 and the corresponding eigenfunction is given in (1), one can establish this statement about g directly. Note that the function g is discontinuous at x = 0. In a forthcoming report we shall show that this situation is typical. In other words we shall show that if K belongs to a gentle class, "it satisfies a local Hölder condition with exponent greater than 1/2", and the Friedrichs' equation fails to admit a continuous solution R, then M+K has a discontinuous point-eigenfunction.

Appendix III.

The behavior of the Hilbert transform at infinity.

Let the function f satisfy an ordinary Hölder condition in $[-\infty, +\infty]$, and be square integrable. Then we maintain that its Hilbert transform, defined by

$$Hf(x) = \int_{-\infty}^{\infty} \frac{f(y)}{y-x}\,dy \quad ,$$

tends to zero as $|x| \longrightarrow \infty$.

First let us notice that

$$Hf(x) = \lim_{\delta \to 0} \int_{|x-y| \geq \delta} \frac{f(y)}{y-x}\,dy$$

and this limit holds uniformly in x. Now clearly

$$\left|\int_{|y-x| \geq \frac{x}{2}} \frac{f(y)}{y-x}\,dy\right| \leq \left(\int_{-\infty}^{\infty} f(y)\overline{f(y)}\,dy\right)^{1/2} \left(\int_{\frac{x}{2}}^{\frac{3x}{2}} \frac{dy}{y^2}\right)^{1/2} = o(1)$$

and

$$\left|\int_{\frac{x}{2} \geq |y-x| \geq \delta} \frac{f(y)}{y-x}\,dy\right| \leq \left(\int_{\frac{x}{2}}^{\frac{3x}{2}} f(y)\overline{f(y)}\,dy\right)^{1/2} \left(\int_{|y| \geq \delta} \frac{dy}{y^2}\right)^{1/2} = o(1)$$

which establishes our claim.

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